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Escape of Photons from an Infinite Cylinder in Brans–Dicke Theory of Gravitation

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Solutions for a cylindrically symmetric static gravitational field in Brans–Dicke theory are obtained. It is found that the condition for escape of photons emitted from the surface of the source depends on two parameters, instead of only the mass parameter as in the case of Mardar's solution in Einstein's theory.

1. INTRODUCTION

It was shown previously by Banerjee (1968) that photons emitted from the surface of an infinitely long cylindrical mass distribution, and moving in a plane perpendicular to its axis, are allowed to escape to infinity only when a parameter interpreted as mass per unit length of the cylinder is below a critical limit. It is found in the present note that in a corresponding case of cylindrical gravitational field in Brans–Dicke theory, there will be two parameters instead of one, which are important in determining whether photons emitted from the surface will escape or not.

In Section 2, an exact static exterior solution is obtained for a cylindrically symmetric gravitational field in Brans–Dicke Theory, and from the field equations a relation connecting the scalar field ϕ and g_{00} of the metric is derived. It is, however, exactly the same as was previously found in a more general case of symmetry by Banerjee and Bhattacharya (1979). The solutions thus obtained reduce to those of Mardar (1958) in general relativity when the scalar field is absent.

In Section 3, null geodesics in the $Z=\text{const}$ plane are studied, and it is shown that two factors, (i) the parameter interpreted as the mass per unit lenght, and (ii) the coupling parameter relating g_{00} with the scalar field ϕ, are generally important in determining whether the light rays emitted from the surface of the cylinder will escape to infinity or will turn back. The

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interesting point to note here is that for a limiting value of the coupling parameter mentioned above, all the rays will escape to infinity irrespective of magnitude of the mass parameter.

2. SOLUTIONS OF THE FIELD EQUATIONS

The most general line element for a static cylindrically symmetric field may be written in the form

$$ds^2 = e^{2\lambda}dt^2 - e^{2\mu}(dr^2 + dz^2) - e^{2\nu}d\phi^2 \tag{1}$$

where μ, ν, λ depend on the radial coordinate r above. The field equations in the matter-free region in the Brans–Dicke (1961) theory are

$$R^{\mu}_{\nu} = -\frac{\omega}{\phi^2}\phi^{,\mu}\phi_{,\nu} - \frac{1}{\phi}\phi^{;\mu}_{;\nu} \tag{2}$$

and the wave equation for the scalar field is

$$\phi^{;\mu}_{;\mu} = 0 \tag{3}$$

Equation (3) for the metric (1) reduces to the relation

$$\phi_i = Be^{-(\lambda+\nu)} \tag{4}$$

where B is an arbitrary constant and a subscript i indicates differentiation with respect to the x^i coordinate.

The field equations (2) are now given explicitly in the form

$$\mu_{11} + \nu_{11} + \lambda_{11} + \nu_1^2 + \lambda_1^2 - \mu_1(\lambda_1 + \nu_1) = \frac{\omega\phi_1^2}{\phi^2} - \frac{\phi_{11}}{\phi} + \frac{\mu_1\phi_1}{\phi} \tag{5}$$

$$\mu_{11} + \mu_1(\nu_1 + \lambda_1) = -\frac{\mu_1\phi_1}{\phi} \tag{6}$$

$$\nu_{11} + \nu_1(\nu_1 + \lambda_1) = -\frac{\nu_1\phi_1}{\phi} \tag{7}$$

$$\lambda_{11} + \lambda_1(\nu_1 + \lambda_1) = -\frac{\lambda_1\phi_1}{\phi} \tag{8}$$

From (4) and (8) one can recover the relation

$$\phi = e^{K\lambda}, \qquad \text{i.e., } \phi = (g_{00})^{K/2} \tag{9}$$

for the cylindrical symmetry, where K is a coupling constant.

Solutions of the field equations (5)–(8) may now be given as

$$\begin{aligned} g_{00} &= e^{2\lambda} = r^{2\alpha}, \qquad g_{33} = -e^{2\nu} = -r^{2a\alpha} \\ g_{11} &= g_{22} = -e^{2\mu} = -Ar^{2b\alpha} \end{aligned} \tag{10}$$

where α, A, a, b are constants with the relations

$$\begin{aligned} a &= \frac{1 - K\alpha - \alpha}{\alpha} \\ 2b &= -a + \alpha + K^2\alpha - \omega K^2\alpha - K + \frac{(1 + K\alpha - \alpha)^2}{\alpha} - 1 \end{aligned} \tag{11}$$

In the absence of the scalar field, i.e., when $K=0$, and $\omega=\infty$, the above solutions reduce to those of the well-known cylindrically symmetric exterior Mardar's solution. The constants α and K could be determined by matching with an interior solution. However, in the absence of a suitable interior solution, α could be interpreted as the mass parameter and as in general relativity it is approximately equal to $2M$. The constant K is found by using the surface integral technique in the following way. We have

$$\phi^{;\mu}_{;\mu} = \frac{8\pi T}{2\omega + 3} \tag{12}$$

or

$$\left[\phi^{\mu}(-g)^{1/2}\right]_{,\mu} = \frac{8\pi T(-g)^{1/2}}{2\omega + 3} \tag{13}$$

Integrating over a unit length of a cylinder of radius d containing matter (i.e., source of the source-free Brans–Dicke solution found), we obtain by Gauss's theorem

$$\int_z^{z+1}\int_0^{2\pi}\left[\phi^{\mu}(-g)^{1/2}\right] dz\, d\phi = \int_0^d\int_z^{z+1}\int_0^{2\pi}\frac{8\pi T(-g)^{1/2}}{2\omega + 3}\, dr\, dz\, d\phi \tag{14}$$

Now, we have

$$\phi=(g_{00})^{K/2}=r^{K\alpha}, \qquad g^{11}=-(1/A)r^{-2b\alpha},$$
$$(-g)^{1/2}=Ar^{(1+a+2b)\alpha}$$

For a static dust cylinder the above equation gives

$$K\approx-\frac{2}{2\omega+3} \tag{15}$$

3. ESCAPE OF PHOTONS

The equations of null geodesics in the $Z=$const plane are

$$r^{2\alpha}\dot{t}^2-r^{2a\alpha}\dot{\phi}^2-Ar^{2\alpha}\dot{r}^2=0 \tag{16}$$

and also

$$\dot{t}=\frac{m}{r^{2(2\alpha+K\alpha-1)}}\dot{\phi} \tag{17}$$

where a dot indicates differentiation with respect to a suitable affine parameter. Substitution of (17) in (16) yields

$$\left(\frac{dr}{d\phi}\right)^2=r^{2(2-b\alpha-3\alpha-2K\alpha)}\left[m^2-r^{2a\alpha+4(2\alpha+K\alpha-1)-2\alpha}\right] \tag{18}$$

Following the arguments of Banerjee in the previous paper (1968), one may conclude that in this case photons once emitted from the surface will escape to infinity or in other words will have no turning point subsequently in their paths provided the condition

$$\alpha(K+2)<1 \tag{19}$$

is satisfied. It is interesting to note that unlike in the general relativity case ($2\alpha<1$), there are now two parameters which are important in determining the behavior of light rays.

Case I. When $-3/2<\omega\leqslant-1$, the condition $\alpha(K+2)<1$ is trivially satisfied for any arbitrary finite value of α, however large it might be, i.e., all the rays emitted from the surface of the cylinder will escape and none of them will be trapped in its course.

Case II. For any positive value of ω the required mass for escape is less than the corresponding case in general relativity.

For circular geodesic motion $dr/ds = dz/ds = 0$ and so

$$r^{2\alpha}\left(\frac{dt}{ds}\right)^2 - r^{2a\alpha}\left(\frac{d\phi}{ds}\right)^2 = \Omega \tag{20}$$

where $\Omega = \pm 1$ for timelike and spacelike geodesics and $\Omega = 0$ for null geodesics.

Again

$$\left(\frac{dt}{ds}\right)^2 = ar^{2\alpha(a-1)}\left(\frac{d\phi}{ds}\right)^2 \tag{21}$$

In view of (20) and (21) are obtained conditions that $\alpha(K+2)$ is greater than, equal to, or less than 1 for all the circular tracks to be timelike, null, or spacelike in the cylindrically symmetric gravitation field in Brans–Dicke theory.

REFERENCES

Banerjee, A. (1968). *Journal of Physics A*, **1**, 495.
Banerjee, A. and Bhattacharya, D. (1979). *Journal of Mathematical Physics*, **20**, 1908.
Mardar, L. (1958), *Proceedings of the Royal Society Series A*, **244**, 524.
Brans, C., and Dicke, R. H. (1961), *Physical Review*, **124**, 925.